Problem 1

Part (a)

Assume that the budget constraint holds with equality. Denote $\alpha c_{t-1} := h_t$ (habit). Then we can write this problem as

$$\sup_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(Ak_t - k_{t+1} - h_t\right) \; s.t.$$

$$k_{t+1} \in (\alpha k, Ak_t - h_t),\ \forall t \geq 0 \tag{1}$$

$$h_{t+1} = \alpha(Ak_t - k_{t+1}),\ \forall t \geq 0$$

$$k_0, h_0 \,\text{given}$$

Note that condition (1), guarantees consumption levels such that $c_t - \alpha c_{t-1} \geq 0,\ \forall t \geq 0$. Because:

$$Ak_0 = \left\{\begin{array}{l} k_1 = \left\{\begin{array}{l} \frac{k_2}{A} = \left\{\begin{array}{l} \frac{k_3}{A^2} \\ \\ \frac{c_2}{A^2} > \frac{\alpha c_1}{A^2} > \frac{\alpha^3 c_{-1}}{A^2} \end{array}\right. \\ \\ \frac{c_1}{A} > \frac{\alpha c_0}{A} > \frac{\alpha^2 c_{-1}}{A} \end{array}\right. \\ \\ c_0 > \alpha c_{-1} \end{array}\right. \qquad ... \to Ak_0 > \frac{\alpha}{1-\frac{\alpha}{A}} c_{-1}$$

$$\to (A-\alpha)k_0 > \alpha c_{-1} = h_0 \to (A-\alpha)k_{t+1} > h_{t+1} = \alpha(Ak_t - k_{t+1}) \to k_{t+1} > \alpha k_t,\ \forall t.$$

Part (b)

Writing the problem in recursive form:

$$V(k,h) = \max_{k'} \left[\ln(Ak - k' - h) + \beta V(k', \alpha(Ak - k'))\right] \; s.t.$$

$$k' \in (\alpha k, Ak - h) \tag{2}$$

Note that we cannot use the standard tools from Stokey Lucas to prove that the solution to this functional equation exists. Because here the return function ($\log(Ak - k' - h)$) is unbounded. More importantly, the domain of the state variables, k and h, is unbounded. If this domain and $\Gamma(k)$ were compact, we could say that the return function is continuous and therefore it is bounded on the compact domain. But this is not the case here.

Part (c)

To solve the functional equation, we first make a guess: $V(k,h) = B\ln(Ck-h)+F$. Next, we proceed to verify this:

$$V(k,h) = \max_{k'} \left[\log\left(Ak-k'-h\right) + \beta\left(B\ln(Ck'-h)+F\right)\right] \ s.t.$$
$$k' \in (\alpha k, Ak-h)$$

Taking the first order condition

$$\frac{1}{Ak-k'-h} = \beta B\frac{C+\alpha}{Ck'-\alpha(Ak-k')} \rightarrow$$
$$\left[C+\alpha+\beta B(C+\alpha)\right]k' = \left[\beta B(C+\alpha)A + A\alpha\right]k - \left[\beta B(C+\alpha)\right]h \rightarrow$$
$$k' = \lambda k - \lambda' h$$

Where $\lambda = \frac{\beta B(C+\alpha)A+A\alpha}{(C+\alpha)(1+\beta B)}$ and $\lambda' = \frac{\beta B}{1+\beta B}$. Substituting this back into the Bellman equation,

$$V(k,h) = \log\left((A-\lambda)k-(1-\lambda')h\right) + \beta\left[B\ln\left((C\lambda-\alpha(A-\lambda))k-(C+\alpha)\lambda' h\right)+F\right]$$

We need

$$\gamma(A-\lambda) = C\lambda - \alpha(A-\lambda)$$
$$\gamma\left(1-\lambda'\right) = (C+\alpha)\lambda'$$
$$1+\beta B = B$$

to hold for some $\gamma \geq 0$. Note that we need not have $A-\lambda = C\lambda-\alpha(A-\lambda)$ (i.e. we need not necessarily $\gamma = 1$).

From $1+\beta B = B$ we get that $B = \frac{1}{1-\beta}$. Using this we get $\lambda' = \beta$ and $C = \gamma\frac{1-\beta}{\beta} - \alpha$. Plugging λ' and simplifying $A-\lambda$ we get

$$\begin{aligned} V(k,h) &= \log\left(\left(\frac{AC\beta}{\gamma}\right)k-(1-\beta)h\right) + \beta\left[B\ln\left(\left(\frac{AC\beta}{\gamma^2}\right)k - \frac{(1-\beta)}{\gamma}h\right)+F\right] \\ &= \log\left(\left(\frac{AC\beta}{\gamma(1-\beta)}\right)k-h\right) + \log(1-\beta) + \beta\left[B\ln\left(\left(\frac{AC\beta}{\gamma(1-\beta)}\right)k-h\right) + B\log(\frac{1-\beta}{\gamma}) + F\right] \end{aligned}$$

We can now solve for γ and C:

$$C = \frac{AC\beta}{\gamma(1-\beta)} \rightarrow \gamma = \frac{A\beta}{1-\beta}$$
$$C = \gamma\frac{1-\beta}{\beta} - \alpha = A - \alpha$$

Finally, we can solve for F:

$$V(k,h) = \frac{1}{1-\beta}\big(\log\left[(A-\alpha)\,k-h\right]\big) + \log(1-\beta) - B\log(A\beta) + \beta F \rightarrow$$

$$F = \log(1-\beta) + B\log(A\beta) + \beta F \rightarrow F(1-\beta) = \log(1-\beta) - B\log(A\beta)$$

$$V(k,h) = \frac{1}{1-\beta}\big(\log\left[(A-\alpha)\,k-h\right]\big) + \frac{1}{1-\beta}\big(\log(1-\beta) - \frac{1}{1-\beta}\log(A\beta)\big)$$

Therefore, our guess is verified, and policy function and habit stock can be written as

$$k' = (A\beta + (1-\beta)\alpha)k - \beta h \tag{3}$$

$$h' = \alpha(A-\alpha)(1-\beta)k - \alpha\beta h \tag{4}$$

Note that the value function we derived can be written as

$$V(k,h) = \frac{\ln k}{1-\beta} + V(1,\hat{h}); \qquad \hat{h} = \frac{h}{k}$$

Hence, we can simplify the value function to having only one state variable $\hat{h}$ and directly solve $V(1,\hat{h})$.

Part (d)

Now we need to determine the domain of the new state variable $\hat{h}$. Remember that since $k=1$ in $V(1,\hat{h})$ we have:

$$\hat{h}' = \frac{h'}{k'} = \frac{\alpha(A-k')}{k'} \rightarrow \hat{h}' = \frac{\alpha A}{k'} - \alpha$$

Since $k' \in (\alpha, A-\hat{h})$, we have that $\hat{h}' \in (\frac{\alpha\hat{h}}{A-\hat{h}}, A-\alpha)$. Next, note that from the policy and habit functions, (3) and (4), which I derived in part (c), we can derive $\hat{h}'$, by setting $k=1$:

$$\hat{h}' = \frac{\alpha(A-\alpha)(1-\beta) + \alpha\beta\hat{h}}{A\beta + (1-\beta)\alpha - \beta\hat{h}}$$

First, observe that

$$\frac{d\hat{h}'}{d\hat{h}} = \frac{\alpha\beta\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right) + \beta\left(\alpha(A-\alpha)(1-\beta) + \alpha\beta\hat{h}\right)}{\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right)^2} = \frac{\alpha\beta\left(A\beta + \alpha(1-\beta) + (A-\alpha)(1-\beta)\right)}{\left(A\beta + (1-\beta)\alpha - \beta\hat{h}\right)^2} > 0.$$

Showing that the policy function $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$. Next, note that we can write the equation

$$\hat{h}' = \hat{h} \rightarrow \frac{\alpha(A-\alpha)(1-\beta) + \alpha\beta\hat{h}}{A\beta + (1-\beta)\alpha - \beta\hat{h}} = \hat{h}$$

As

$$\left(\hat{h} - (A-\alpha)\right)\left(\hat{h} - \frac{\alpha(1-\beta)}{\beta}\right) = 0$$

Hence, for $0 < \hat{h} < \frac{\alpha(1-\beta)}{\beta}$, we have $\hat{h} < \hat{h}'$, and since I have already shown that $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$, we get that the policy sequence is increasing in $\hat{h} \in (0, \frac{\alpha(1-\beta)}{\beta})$.
On the other hand if $\frac{\alpha(1-\beta)}{\beta} < \hat{h} < A - \alpha$, we get that $\hat{h}' < \hat{h}$, and again since $\hat{h}'(\hat{h})$ is increasing in $\hat{h}$, policy sequence is decreasing in $\hat{h} \in (\frac{\alpha(1-\beta)}{\beta}, A-\alpha)$.. Therefore, for any $\hat{h} \in (0, A-\alpha)$ the additional assumptions needed for theorems 4.3 and 4.5 of SLP are satisfied. Hence, we can be sure that this value function is the unique function satisfying the functional equation that satisfies those additional assumptions.

Part (e)

Take the policy function and habit stock

$$k' = (A\beta + (1-\beta)\alpha)k - \beta h$$
$$h' = \alpha(A-\alpha)(1-\beta)k - \alpha\beta h$$

Divide k' and h' by k and h respectively

$$\frac{k'}{k} = (A\beta + (1-\beta)\alpha) - \beta\frac{h}{k}$$
$$\frac{h'}{h} = \alpha(A-\alpha)(1-\beta)\frac{k}{h} - \alpha\beta$$

I have already shown in part (d) the steady state value for $\frac{h}{k} = \frac{\alpha(1-\beta)}{\beta}$, and I have reasoned why $\hat{h}$ converges to that. Therefore I only need to characterize the BGP where $\hat{h}$ reaches its steady state value and capital, k, and habit stock, h, grow at a constant rate

$$\frac{k'}{k} = (A\beta + (1-\beta)\alpha) - \beta\frac{h}{k} = A\beta$$
$$\frac{h'}{h} = \frac{c'}{c} = \alpha(A-\alpha)(1-\beta)\frac{k}{h} - \alpha\beta = A\beta$$

Hence, on BGP the growth rate of capital and consumption is equal to $A\beta$.

Part (f)

Starting from an initial value $0 < \hat{h} < \frac{\alpha(1-\beta)}{\beta}$, I proved that we have $\hat{h} < \hat{h}'$. By $\frac{dh'}{d\hat{h}} > 0$, we have that the policy sequence increases towards the steady state point.

Similarly starting from an initial value $\frac{\alpha(1-\beta)}{\beta} < \hat{h} < A-\alpha$, we have $\hat{h}' < \hat{h}$, which means the policy sequence will decrease towards its steady state. This means if the economy has a low habit stock to capital ratio, agents consume more and buid up a higher habit stock. On the other hand, if the habit stock is relatively large with respect to the capital, the economy will move towards investing more and lowering the ratio of habit stock relative to capital.

Problem 2

Part (a)

$$\max_{\{c_t\}_t} W(\mathbf{c}) \ s.t.$$
$$c_t + k_{t+1} - (1-\delta)k_t \leq f(k_t)$$
$$k_0 \text{ given}$$

Denote by $g(k, k') := f(k) + (1-\delta)k - k'$. The associated Bellman equation is

$$V(k) = \max_{k'} G(g(k, k'), V(k')) \ s.t.$$
$$k' \in [(1-\delta)k, Ak + (1-\delta)k]$$

Part (b)

- G_1: continuous and bounded
- G_2: concave (not necessary for this exercise)
- G_3: $G(0, 0) = 0$
- G_4: $(x, z) \leq (x, z)$ and $(x', z') \neq (x, z)$ implies $G(x, z) < G(x', z')$
  and for some $0 \leq \beta < 1$,
- G_5: $|G(x, z) - g(x, z')| \leq \beta|z - z'|$ for all $x \in R_+^m$ and all $z, z' \in R_+$.

Now I will show that given this set of assumptions, we can prove that there is exactly one continuous bounded function V satisfying

$$V(k) = \max_{k'} G(g(k, k'), V(k')) \ s.t.$$
$$k' \in [(1-\delta)k, Ak + (1-\delta)k]$$

Let C be the Banach space of continuous bounded functions f with norm

$$||f|| = sup_k |f(k)|$$

Let T be the operator on C defined by $(Tf)(k) = \max_{k'} G(g(k,k'), f(k'))$. The set $[(1-\delta)k, Ak + (1-\delta)k]$ is compact and the function to be maximized in applying T is continuous, so Tf is well defined for each $f \in C$. Tf is continuous and evidently bounded, so that $T : C \rightarrow C$. Next, we show that T is a contraction mapping, by showing that it satisfies the Blackwell's sufficient conditions

$$f \leq f' \rightarrow Tf \leq Tf', \qquad \text{for all } f, f' \in C$$

and for some $\beta \in [0,1)$

$$T(f + a) \leq Tf + \beta a, \qquad \text{for all } f \in C \text{ and all } a > 0$$

Note that the monotonicity condition follows from G_4, i.e. $(x,z) \leq (x',z')$ and $(x',z') \neq (x,z)$ implies $G(x,z) < G(x',z')$. Since we have $f(k) + (1-\delta)k - k' \leq f'(k) + (1-\delta)k - k'$ and $f(k') \leq f'(k')$, we get that $G(g(k,k'), f(k')) \leq G(g'(k,k'), f'(k'))$.
Next since $[(1-\delta)k, f(k) + (1-\delta)k] \subset [(1-\delta)k, f'(k) + (1-\delta)k]$, we get that $Tf' \geq Tf$.
The second property, discounting, follows directly from G_4 and G_5, i.e. $|G(x,z) - G(x,z')| \leq \beta|z - z'|$ for all $x \in R_+^m$ and all $z, z' \in R_+$.

Let $k'^* = armax_{k'} G(g(k,k'), f(k') + a)$. By G_5 we have

$$|G(g(k,k'^*), f(k'^*) + a) - G(g(k,k'^*), f(k'^*))| \leq \beta a$$

Note that by G_4, $G(g(k,k'^*), f(k'^*)) \leq G(g(k,k'^*), f(k'^*) + a)$. Therefore we get that

$$G(g(k,k'^*), f(k'^*) + a) \leq G(g(k,k'^*), f(k'^*)) + \beta a \leq (Tf)(k) + \beta a.$$

Hence, by contraction mapping theorem, T has a unique fixed point, which is the solution to our functional equation. Epstein Zin preferences are an example of recursive preferences. Also the functional equation given in the problem 3 of the previous problem set is another example. Remember that there I proved continuity, boundedness, monotonicity, and discounting properties.

Part (c)

Taking the first order condition

$$\frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k') = 0$$

Envelope condition

$$V'(k) = \frac{dG}{dx}.\frac{dg}{dk} = \frac{dG}{dx}.(f'(k) + 1 - \delta)$$

In steady state $k' = k$

$$\frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k') = \frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.V'(k) = \frac{dG}{dx}\frac{dg}{dk'} + \frac{dG}{dz}.\frac{dG}{dx}.(f'(k) + 1 - \delta) = 0 \rightarrow$$

$$\frac{dg}{dk'} + \frac{dG}{dz}.(f'(k) + 1 - \delta) = 0 \rightarrow \frac{dG}{dz}.(f'(k) + 1 - \delta) = 1.$$

Problem 3

This problem is studied in Greenwood, Hercowitz and Per Krusell (1997) - see the paper if you want to learn more about it.

Part (a)

For this problem I ignore profits (which will be zero at the CE). I assume that households have a fixed endowment of leisure $\bar{e} = 1$ and that leisure is unvalued. A competitive equilibrium is prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ and allocations $\{c_t^i, x_t^i, k_t^i, y_t^f, n_t^f, k_t^f\}_{t=0}^{\infty}$ such that

1. Households solve the problem

$$\max_{\{c_t^i, x_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \ s.t.$$

$$p_t(c_t^i + x_t^i) \le r_t k_t^i + w_t \frac{1}{I}$$

$$k_{t+1} = (1-\delta)k_t + q_t x_t$$

$$q_{t+1} = (1 + g_q)q_t$$

$$k_{t+1} \ge 0 \ \forall t$$

$$k_0 \text{ given}$$

2. The representative firm maximizes profits:

$$\max_{y_t^f, k_t^f} p_t y_t^f - r_t k_t^f - w_t n_t^f \ s.t.$$

$$y_t^f = z(k_t^f)^{\alpha}(n_t^f)^{1-\alpha}$$

3. Markets clear:

$$\sum_i \left(c_t^i + x_t^i\right) = y_t^f$$
$$1 = n_t^f$$
$$\sum_i k_t^i = k_t^f \qquad \forall\ i$$

Part (b)

First note that on any balanced growth path, both sides of the investment equation

$$k_{t+1} - (1-\delta)k_t = x_t q_t$$

have to grow at the same rate. If $(1+g_k)$ is the growth rate of capital, and $(1+g_x)$ the growth rate of investment, then we have

$$1 + g_k = (1+g_x)(1+g_q)$$

Additionally, from the production function we must have that at any BGP,

$$(1+g_y) = (1+g_k)^{\alpha}$$

Since $\alpha < 1$, it is clearly the case that $g_k > g_y$. Supposing that $g = g_y = g_x$ (i.e. investment and output [and consumption] all grow at the same rate), we have

$$1 + g_k = (1+g)(1+g_q)$$
$$1 + g = (1+g_k)^{\alpha}$$

and hence

$$1 + g = (1+g_q)^{\frac{\alpha}{1-\alpha}}$$
$$1 + g_k = (1+g_q)^{\frac{1}{1-\alpha}} \tag{1}$$

Part (c)

To write the planner's problem recursively, we need to first put the model in terms of a steady state, which means dividing out by the growth rates. Denote the following:

$$\hat{c}_t = c_t/(1+g)^t$$
$$\hat{x}_t = x_t/(1+g)^t$$
$$\hat{u}_t = y_t/(1+g)^t$$
$$\hat{k}_t = k_t/(1+g_k)^t$$
$$\hat{q}_t = q_t/(1+g_q)^t$$

Then

$$\begin{aligned}\hat{x}_t &= \frac{k_{t+1}-(1-\delta)k_t}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}q_t} \\ &= \frac{(1+g_q)^{\frac{t+1}{1-\alpha}}\hat{k}_{t+1}-(1+g_q)^{\frac{t}{1-\alpha}}(1-\delta)\hat{k}_t}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}(1+g_q)^t\hat{q}_t} \\ &= \frac{(1+g_q)^{\frac{1}{1-\alpha}}\hat{k}_{t+1}-(1-\delta)\hat{k}_t}{\hat{q}_t} \\ &= \frac{(1+g_q)^{\frac{1}{1-\alpha}}\hat{k}_{t+1}-(1-\delta)\hat{k}_t}{\hat{q}}\end{aligned}$$

where the last equality follows from the fact that $\hat{q}_{t+1}/\hat{q}_t = 1$. And in yet another example of the wonders of Cobb-Douglas:

$$\begin{aligned}\hat{y}_t &= \frac{zk_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= \frac{z(1+g_q)^{\frac{t\alpha}{1-\alpha}}\hat{k}_t^\alpha}{(1+g_q)^{\frac{t\alpha}{1-\alpha}}} \\ &= z\hat{k}_t^\alpha\end{aligned}$$

We therefore have the planner's problem

$$\max_{\{\hat{c}_t,\hat{k}_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(\hat{c}_t)\ s.t.$$
$$\hat{c}_t + \frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}}k_{t+1} = \frac{1-\delta}{\hat{q}}\hat{k}_t + z\hat{k}_t^\alpha$$

Substituting the feasibility constraint into the utility function and taking the FOC to get the Euler equation:

$$\frac{(1+g_q)^{\frac{1}{1-\alpha}}}{\hat{q}} u_{c,t} = \beta u_{c,t+1} \left(z\alpha \hat{k}_{t+1}^{\alpha-1} + \frac{1-\delta}{\hat{q}} \right)$$

and rewriting,

$$\frac{u_{c,t}}{u_{c,t+1}} = \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(\hat{q} z\alpha \hat{k}_{t+1}^{\alpha-1} + 1 - \delta \right) \tag{2}$$

If the economy starts with a low level of capital, the RHS of (2) is greater than 1 - in which case the left-hand side is also greater than 1, which implies that consumption is growing (and the economy accumulating capital) given the standard assumption that u is strictly concave. On the other hand, as capital goes towards infinity, we have

$$\lim_{\hat{k}\to\infty} \frac{\beta}{(1+g_q)^{\frac{1}{1-\alpha}}} \left(\hat{q} z\alpha \hat{k}_{t+1}^{\alpha-1} + 1 - \delta \right) = \frac{\beta(1-\delta)}{(1+g_q)^{\frac{1}{1-\alpha}}}$$

which is clearly less than 1, indicating that investment is negative and consumption is declining over time. Hence the economy will eventually converge to a steady state where the LHS of (2) is equal to 1, and at this level we will have

$$\hat{k} = \left(\frac{\beta^{-1}(1+g_q)^{\frac{1}{1-\alpha}} - (1-\delta)}{\hat{q} z\alpha} \right)^{\frac{1}{\alpha-1}} \tag{3}$$

Part (d)

The rental price of capital will be the marginal return of capital:

$$\begin{aligned} r_t &= z\alpha k_t^{\alpha-1} \\ &= z\alpha \left(\frac{1}{k_t} \right)^{1-\alpha} \end{aligned}$$

showing that the ratio of labor to capital is decreasing, and hence the rental rate of capital is decreasing. At the BGP we can write the rental rate explicitly as

$$\begin{aligned} r_t &= z\alpha \left(\frac{1}{k_0(1+g_q)^{\frac{t}{1-\alpha}}} \right)^{1-\alpha} \\ &= z\alpha k_0^{\alpha-1}(1+g_q)^{-t} \end{aligned} \tag{4}$$

Now at any equilibrium, we must have the return to saving equal to the return to investing. From (2) above, we have

$$\begin{aligned}\frac{u_{c,t}}{\beta u_{c,t+1}} &= i_t \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\hat{q}z\alpha\hat{k}_{t+1}^{\alpha-1}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(\frac{q_t}{(1+g_q)^t}z\alpha\left(\frac{k_t}{(1+g_q)^{\frac{t}{1-\alpha}}}\right)^{\alpha-1}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(q_0(1+g_q)^t z\alpha k_0^{\alpha-1}(1+g_q)^{-t}+1-\delta\right) \\ &= \frac{1}{(1+g_q)^{\frac{1}{1-\alpha}}}\left(q_0 z\alpha k_0^{\alpha-1}+1-\delta\right) \qquad (5)\end{aligned}$$

showing that the interest rate is constant at the BGP. Basically: the return to capital *per unit* is falling, but because units of capital become cheaper over time relative to consumption, the return to investment (and hence the return to saving) is constant.

Part (e)

Repeating the steps in part (b), we arrive at the system of equations:

$$\begin{aligned}(1+g_e) &= (1+g)(1+g_q) \\ (1+g) &= (1+g_z)(1+g_e)^{\alpha_e}(1+g)^{\alpha_s}\end{aligned}$$

with solutions

$$\begin{aligned}1+g_e &= (1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}} \\ 1+g &= (1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\end{aligned}$$

Hence equipment grows at a faster rate than structures, and so the ratio of equipment to structures is growing over time. Now we wish to examine the rental rate and investment rate. Using the same notation as before, we have

$$\begin{aligned}
\hat{x}_t &= \frac{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1}\hat{k}^e_{t+1} - (1-\delta_e)\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{k}^e_t}{\left((1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}(1+g_q)^t\hat{q}_t} \\
&= \frac{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t+1}\hat{k}^e_{t+1} - (1-\delta_e)\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{k}^e_t}{\left((1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\right)^{t}\hat{q}_t} \\
&= \frac{(1+g_q)^{\frac{1-\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\hat{k}_{t+1} - (1-\delta)\hat{k}_t}{\hat{q}} \\
\hat{i}_t &= (1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}\hat{k}^s_{t+1} + (1-\delta_s)\hat{k}^s_t
\end{aligned}$$

And if we follow the same steps as in part (d), we obtain the rental rate

$$\begin{aligned}
r_t &= z_0(1+g_z)^t\alpha\left(\frac{1}{k^e_0(1+g_q)^{\frac{t-t\alpha_s}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}}\right)^{1-\alpha_e}\left(k^s_0(1+g_q)^{\frac{t\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{t}{1-\alpha_s-\alpha_e}}\right)^{\alpha_s} \\
&= z_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s}(1+g_q)^{-t(1-\alpha_s-\alpha_e)} \qquad (6)
\end{aligned}$$

which is again decreasing over time. Likewise the investment rate will be

$$\begin{aligned}
i_t &= \frac{\beta}{(1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}}\left(\hat{q}\hat{z}_t\alpha_e(\hat{k}^e_t)^{\alpha_e-1}(\hat{k}^s_t)^{\alpha_s} + 1 - \delta\right) \\
&= \frac{\beta}{(1+g_q)^{\frac{\alpha_e}{1-\alpha_s-\alpha_e}}(1+g_z)^{\frac{1}{1-\alpha_s-\alpha_e}}}\left(q_0\alpha_e(k^e_0)^{\alpha_e-1}(k^s_0)^{\alpha_s} + 1 - \delta\right) \qquad (7)
\end{aligned}$$

which remains constant as before.

Problem 4

Structural transformation refers to the process of reallocation of economic activity across sectors as the economy experiences growth. This is one feature of modern economic growth that is missing from the one-sector growth model. Allowing for this extension can help us determine whether reallocation across sectors arises as an efficient equilibrium outcome. In this problem, we study conditions under which structural transformation and balanced growth are simultaneously possible.

Part (a)

Let $p_{1,t}$ and $p_{2,t}$ denote the prices of good 1 and good 2, respectively. Let $x_{1,t}$ denote the investment in good 1. A competitive equilibrium is a sequence of allocations $\{c_{1,t}, c_{2,t}, k_{t+1}\}$, output

$\{y_{1,t}, k_{1,t}, n_{1,t}\}$, $\{y_{2,t}, k_{2,t}, n_{2,t}\}$ and prices $\{p_{1,t}, p_{2,t}, r_t, w_t\}$ such that:

1. Households maximize utility taking prices as given:

$$\max_{\{c_{1,t},c_{2,t},k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [a \log c_{1,t} + (1-a) \log c_{2,t}] \; s.t.$$
$$\sum_{t=0}^{\infty} p_{1,t}(c_{1,t} + x_{1,t}) + p_{2,t}c_{2,t} \leq \sum_{t=0}^{\infty} r_t k_t + w_t n_t$$
$$k_{t+1} = (1-\delta)k_t + x_{1,t}, \; \forall t \geq 0$$
$$k_{t+1}, c_t \geq 0; \; \forall t \geq 0$$
$$k_0 \text{ given}$$

2. Firms maximize profits taking prices as given:

$$\text{Good 1 Sector:} \quad \max_{n_{1,t},k_{1,t}} p_{1,t} k_{1,t}^{\alpha} (A_{1,t} n_{1,t})^{1-\alpha} - w_t n_{1,t} - r_t k_{1,t}$$
$$\text{Good 2 Sector:} \quad \max_{n_{2,t},k_{2,t}} p_{2,t} k_{2,t}^{\alpha} (A_{2,t} n_{2,t})^{1-\alpha} - w_t n_{2,t} - r_t k_{2,t}$$

3. Markets Clear:

$$\sum_i [c_{1,t} + x_{1,t}] = k_{1,t}^{\alpha} (A_{1,t} n_{1,t})^{1-\alpha}$$
$$\sum_i c_{2,t} = k_{2,t}^{\alpha} (A_{2,t} n_{2,t})^{1-\alpha}$$
$$\sum_i k_t = k_{1,t} + k_{2,t}$$
$$\sum_i n_t = n_{1,t} + n_{2,t} = 1$$

CE First Order Conditions

From HH Side:

$$\begin{cases} [c_{1,t}] : \; \frac{a}{c_{1,t}} - \lambda p_{1,t} = 0 \\ [c_{2,t}] : \; \frac{1-a}{c_{2,t}} - \lambda p_{2,t} = 0 \end{cases} \Rightarrow \frac{p_{1,t}}{p_{2,t}} \frac{c_{1,t}}{c_{2,t}} = \frac{a}{1-a} \tag{1}$$

From Firm Side:

$$\begin{cases} [k_{1,t}]: \; r_t = p_{1,t}\alpha\left(\frac{k_{1,t}}{n_{1,t}}\right)^{\alpha-1} A_{1,t}^{1-\alpha} \\ [n_{1,t}]: \; w_t = p_{1,t}(1-\alpha)\left(\frac{k_{1,t}}{n_{1,t}}\right)^{\alpha} A_{1,t}^{1-\alpha} \end{cases} \Rightarrow K_t = \frac{k_{1,t}}{n_{1,t}} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t} \tag{2}$$

$$\begin{cases} [k_{2,t}]: \; r_t = p_{2,t}\alpha\left(\frac{k_{2,t}}{n_{2,t}}\right)^{\alpha-1} A_{2,t}^{1-\alpha} \\ [n_{2,t}]: \; w_t = p_{2,t}(1-\alpha)\left(\frac{k_{2,t}}{n_{2,t}}\right)^{\alpha} A_{2,t}^{1-\alpha} \end{cases} \Rightarrow K_t = \frac{k_{2,t}}{n_{2,t}} = \frac{\alpha}{1-\alpha}\frac{w_t}{r_t} \tag{3}$$

From (2) and (3), we observe:

1. Capital-to-labor ratios are equalized across sectors at every period:

$$\frac{k_{1,t}}{n_{1,t}} = \frac{k_{2,t}}{n_{2,t}} = \frac{k_{1,t}}{n_{1,t}}n_{1,t} + \frac{k_{2,t}}{n_{2,t}}n_{2,t} = K_t(n_{1,t}+n_{2,t}) = K_t$$

2. Relative prices across sectors are pinned down by technology

$$\frac{p_{2,t}}{p_{1,t}} = \left(\frac{A_{1,t}}{A_{2,t}}\right)^{1-\alpha} = P_t$$

From (1) and the above observation about relative prices, we have:

$$\frac{c_{2,t}}{c_{1,t}} = \frac{1-a}{a}\left(\frac{A_{2,t}}{A_{1,t}}\right)^{1-\alpha}$$

We can use P_t and K_t to show that the model aggregates on the production side:

$$\begin{aligned}(c_{1,t}+x_{1,t}) + P_t(c_{2,t}) &= k_{1,t}^{\alpha}(A_{1,t}n_{1,t})^{1-\alpha} + P_t k_{2,t}^{\alpha}(A_{2,t}n_{2,t})^{1-\alpha} \\ &= \left(\frac{k_{1,t}}{n_{1,t}}\right)^{\alpha} A_{1,t}^{1-\alpha} n_{1,t} + P_t A_{2,t}^{1-\alpha}\left(\frac{k_{2,t}}{n_{2,t}}\right)^{\alpha} n_{2,t} \\ &= \left(\frac{k_{1,t}}{n_{1,t}}\right)^{\alpha} A_{1,t}^{1-\alpha} n_{1,t} + A_{1,t}^{1-\alpha}\left(\frac{k_{2,t}}{n_{2,t}}\right)^{\alpha} n_{2,t} \\ &= K_t^{\alpha} A_{1,t}^{1-\alpha}(n_{1,t}+n_{2,t}) \\ &= K_t^{\alpha} A_{1,t}^{1-\alpha}\end{aligned}$$

This suggests that we can write this in recursive form:

$$V(k) = \max_{k'} \log(K_t^{\alpha} A_{1,t}^{1-\alpha} - k_{t+1} + (1-\delta)k_t) + \beta V(k')$$

Euler Equation:

$$\frac{c_{t+1}}{c_t} = \beta[\alpha K_{t+1}^{\alpha-1} A_{1,t+1}^{1-\alpha} + (1-\delta)]$$

Part (b)

The solution to the *Competitive Equilibrium* defined above coincides with the *Pareto Optimal* solution. The problem becomes:

$$\max_{\{c_{1,t}, c_{2,t}, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [a \log c_{1,t} + (1-a) \log c_{2,t}] \ s.t.$$

$$c_{2,t} = k_{2,t}^{\alpha}(A_{2,t} n_{2,t})^{1-\alpha} \qquad (\lambda_{1,t})$$

$$k_{t+1} = k_{1,t}^{\alpha}(A_{1,t} n_{1,t})^{1-\alpha} + (1-\delta)k_t - c_{1,t} \qquad (\lambda_{2,t})$$

$$n_{1,t} + n_{2,t} = 1$$

$$k_0 \text{ given}$$

SP First Order Conditions

$$[c_{1,t}]: \ \frac{\beta^t a}{c_{1,t}} - \lambda_{1,t} = 0 \tag{4}$$

$$[c_{2,t}]: \ \frac{\beta^t a}{c_{2,t}} - \lambda_{2,t} = 0 \tag{5}$$

$$[k_{1,t+1}]: \ -\lambda_{1,t} + \lambda_{1,t+1}(1-\delta) + \lambda_{1,t+1}\alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1} n_{1,t+1})^{1-\alpha}) = 0 \tag{6}$$

$$[k_{2,t+1}]: \ -\lambda_{1,t} + \lambda_{1,t+1}(1-\delta) + \lambda_{2,t+1}(\alpha k_{2,t+1}^{\alpha-1}(A_{2,t+1} n_{2,t+1})^{1-\alpha}) = 0 \tag{7}$$

From (4) and (5), we observe:

$$\frac{c_{1,t}}{c_{2,t}} = \frac{a}{1-a}\frac{\lambda_{2,t}}{\lambda_{1,t}}$$

$$\frac{c_{1,t+1}}{c_{1,t}} = \beta \frac{\lambda_{1,t}}{\lambda_{1,t+1}}$$

From (6) and (7), we observe:

$$\frac{\lambda_{1,t+1}}{\lambda_{2,t+1}} = \left(\frac{A_{2,t+1}}{A_{1,t+1}}\right)^{1-\alpha}$$

$$\frac{\lambda_{1,t}}{\lambda_{1,t+1}} = 1 - \delta + \alpha k_{1,t+1}^{\alpha-1}(A_{1,t+1} n_{1,t+1})^{1-\alpha}$$

Iterating this equation back one period and combining with previous equation, we have:

$$\frac{c_{2,t}}{c_{1,t}} = \frac{1-a}{a}\left(\frac{A_{2,t}}{A_{1,t}}\right)^{1-\alpha}$$

From $c_{2,t} = k_{2,t}^{\alpha}(A_{2,t}n_{2,t})^{1-\alpha}$, we have:

$$\frac{c_{2,t}}{n_{2,t}} = \left(\frac{k_{2,t}}{n_{2,t}}\right)^{\alpha} A_{2,t}^{1-\alpha}$$

Part (c)

Under the standard definition of BGP, endogenous variables are constant or grow at constant rates. However, structural transformation requires prices, quantities and employment to change over time due to the reallocation. In this setting, we define a balanced growth path as a path where output, consumption and capital in each sector grow at the same rate.

When $g_1 = g_2 = g$, we have:

$$\frac{A_{1,t+1}}{A_{1,t}} = \frac{A_{2,t+1}}{A_{2,t}} = 1+g \Rightarrow \frac{c_{2,t+1}}{c_{2,t}} = \frac{c_{1,t+1}}{c_{1,t}}$$

We thus conjecture:

$$\frac{c_{1,t+1}}{c_{1,t}} = \beta\frac{\lambda_{1,t}}{\lambda_{1,t+1}} = \beta(1-\delta+\alpha k_{1,t+1}^{\alpha-1}A_{1,t+1}^{1-\alpha}n_{1,t+1}^{1-\alpha}) = 1+\hat{g}$$

$$\Rightarrow \frac{\alpha A_{1,t+1}^{1-\alpha}\left(\frac{n_{1,t+1}}{k_{1,t+1}}\right)^{1-\alpha}}{\alpha A_{1,t}^{1-\alpha}\left(\frac{n_{1,t}}{k_{1,t}}\right)^{1-\alpha}} = 1$$

$$\Rightarrow \frac{k_{1,t+1}}{k_{1,t}}\frac{n_{1,t}}{n_{1,t+1}} = 1+g$$

Note that since $n_{1,t} + n_{2,t} = n_{1,t+1} + n_{2,t+1} = 1$ labor shares cannot be growing on the BGP (you can easily see this by first supposing labor shares and growing at constant (potentially different) rates and then deriving a contradiction). Since $\frac{n_{1,t}}{n_{1,t+1}} = 1$, we have $\frac{k_{1,t+1}}{k_{1,t}} = 1+g$. Remember that since $\frac{k_{1,t}}{n_{1,t}} = \frac{k_{2,t}}{n_{2,t}}$, we have $\frac{k_{1,t+1}}{k_{1,t}}\frac{n_{1,t}}{n_{1,t+1}} = \frac{k_{2,t+1}}{k_{2,t}}\frac{n_{2,t}}{n_{2,t+1}}$, and therefore $\frac{k_{2,t+1}}{k_{2,t}} = 1+g$. Since capital in each sector is growing at the same rate as the labor augmenting productivity, we have

$$\frac{Y_{1,t+1}}{Y_{1,t}} = \left(\frac{k_{1,t+1}}{k_{1,t}}\right)^{\alpha}\left(\frac{A_{1,t+1}}{A_{1,t}}\right)^{1-\alpha} = 1+g$$
$$\frac{Y_{2,t+1}}{Y_{2,t}} = \left(\frac{k_{2,t+1}}{k_{2,t}}\right)^{\alpha}\left(\frac{A_{2,t+1}}{A_{2,t}}\right)^{1-\alpha} = 1+g$$

and therefore

$$\frac{c_{1,t+1}}{c_{1,t}} = \frac{c_{2,t+1}}{c_{2,t}} = \frac{k_{1,t+1}}{k_{1,t}} = \frac{k_{2,t+1}}{k_{2,t}} = \frac{Y_{1,t+1}}{Y_{1,t}} = \frac{Y_{2,t+1}}{Y_{2,t}} = 1+g$$

Part (d)

When $g_1 < g_2$, the sector with higher productivity (i.e., sector 2) produces relatively more output over time. Price changes will be such that the demand for consumption adjusts according to the differential growth rates relative to output changes.

$$\frac{A_{2,t+1}}{A_{1,t+1}} = \left(\frac{A_{2,0}}{A_{1,0}}\right)\left(\frac{1+g_2}{1+g_1}\right)^{t+1}$$

As $t \to \infty$, we have:

$$\frac{c_{1,t}}{c_{2,t}} = \frac{a}{1-a}\left(\frac{A_{1,t}}{A_{2,t}}\right)^{1-\alpha} \to 0$$

Hence in this case, in long run the sector with low productivity growth will go out of the business.

Problem 5

See the solutions manual to Ljungqvist and Sargent (it's easy to find).

Problem 6

Part (a)

For simplicity I assume that labor is inelastically supplied, with aggregate endowment equal to 1. With Pareto weights, we can write the problem as the following:

$$\max_{\{c_t^R, c_t^P, k_{t+1}^R, k_{t+1}^P\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\alpha_R \log(c_t^R - \bar{c}) + \alpha_P \log(c_t^P - \bar{c})\right) \ s.t.$$
$$c_t^R + c_t^P + k_{t+1}^R + k_{t+1}^P \le A(k_t^R + k_t^P)^{\alpha} + (1-\delta)(k_t^R + k_t^P)$$

In any given period, the policymaker will face the static problem of dividing output between the rich and the poor. The solution to this intratemporal problem is given by the FOC's:

$$\begin{aligned}\frac{\beta^t \alpha_R}{c_t^R - \bar{c}} &= \lambda_t \\ &= \frac{\beta^t \alpha_P}{c_t^P - \bar{c}} \\ \implies c_t^P &= \frac{\alpha_R - \alpha_P}{\alpha_R}\bar{c} + \frac{\alpha_P}{\alpha_R} c_t^R\end{aligned}$$

implying that c_P is a convex combination of c_R and $\bar{c}$. Hence we can write the following:

$$\begin{aligned}c_t &= \frac{c_t^R + (\alpha_R - \alpha_P)\bar{c}}{\alpha_R} \\ c_t^R &= \alpha_R c_t - (\alpha_R - \alpha_P)\bar{c} \\ c_t^P &= \alpha_P c_t + (\alpha_R - \alpha_P)\bar{c}\end{aligned}$$

Hence we can write the problem recursively:

$$V(k) = \max_{k'} \left[\alpha_R \log(\alpha_R[Ak^\alpha - k'] + (1-\delta)k_t - 2\alpha_R \bar{c}) + \alpha_P \log(\alpha_P[Ak^\alpha + (1-\delta)k_t - k'] - 2\alpha_P \bar{c}) + \beta V(k')\right]$$

Part (b)

Taking FOC's,

$$\begin{aligned}\frac{\alpha_R + \alpha_P}{Ak^\alpha + (1-\delta)k - k' - 2\bar{c}} &= \beta V'(k') \\ V'(k) &= \frac{(\alpha_R + \alpha_P)[A\alpha K^{\alpha-1} + 1 - \delta]}{Ak^\alpha + (1-\delta)k - k' - 2\bar{c}}\end{aligned}$$

which gives us the Euler equation

$$\frac{1}{c_t - \bar{c}} = \beta \frac{A\alpha K^{\alpha-1} + 1 - \delta}{c_{t+1} - \bar{c}}$$

At the steady state, we will have $c_t = c_{t+1}$ and hence

$$k_{ss} = \left(\frac{1 - \beta(1-\delta)}{\beta A \alpha}\right)^{\frac{1}{\alpha-1}}$$

which does not depend on the welfare weights.

Part (c)

From part (a), we have

$$c_t^R / c_t = \alpha_R - (\alpha_R - \alpha_P)\frac{\bar{c}}{c_t} \tag{1}$$

$$c_t^P / c_t = \alpha_P + (\alpha_R - \alpha_P)\frac{\bar{c}}{c_t} \tag{2}$$

Observe that as c_t becomes very large relative to $\bar{c}$, the right-hand sides of (1) and (2) drop out, and each group's consumption converges to their welfare weights. At the other extreme, where $c_t = 2\bar{c}$, each group consumes 1/2 of aggregate consumption. Hence, a high $\bar{c}$ tends to increase income equality, but it also means that inequality will change as aggregate consumption increases or decreases.

Assuming that $c_{ss} > 2\bar{c}$, in considering the evolution of inequality we need to consider two scenarios:

1. $c_0 > c_{ss}$: in this case, consumption becomes less unequal over time. Essentially, 'luxury' becomes less affordable over time, and consumption for both groups converges towards (though not necessarily to) subsistence levels.

2. $c_0 < c_{ss}$: now consumption becomes more unequal over time. Because growth in consumption is directed entirely in luxury or 'excess' consumption (i.e. consumption greater than $\bar{c}$, and because the wealthy are allowed to consume α_R of this excess consumption, the ratio c_t^R/c_t^p becomes greater over time. At the limit, this ratio converges to α_R/α_P.

And of course in the trivial case where $c_0 = c_{ss}$, inequality doesn't change at all.

Part (d)

First, note that from the agents' problems we have

$$\max_{\{c_t^i, k_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i - \bar{c}) \ s.t.$$

$$c_t, k_{t+1} \geq 0 \ \forall t$$

$$k_0 \text{ given}$$

F.O.C:

$$[c_t^i] : \frac{\beta^t}{c_t^i - \bar{c}} - \lambda_i p_t = 0 \to c_t^i - \bar{c} = \frac{\beta^t}{\lambda_i p_t} \to \frac{c_{t+1}^i - \bar{c}}{c_t^i - \bar{c}} = \beta \frac{p_t}{p_{t+1}} \quad (3)$$

$$\to c_t^R - \bar{c} = \frac{\beta^t}{\lambda_R p_t} = \alpha_R(c_t - 2\bar{c}) = \alpha_R \left(\frac{\beta^t}{p_t} \left[\frac{1}{\lambda_R} + \frac{1}{\lambda_P} \right] \right) \to \alpha_R = \frac{\lambda_P}{\lambda_R + \lambda_P} \quad (4)$$

$$c_t^P - \bar{c} = \frac{\beta^t}{\lambda_P p_t} = \alpha P(c_t - 2\bar{c}) = \alpha_P \left(\frac{\beta^t}{p_t} \left[\frac{1}{\lambda_R} + \frac{1}{\lambda_P} \right] \right) \to \alpha_P = \frac{\lambda_R}{\lambda_R + \lambda_P} \quad (5)$$

$$[k_{t+1}^i] : p_t - p_{t+1}(1-\delta) - r_{t+1} = 0 \to p_t = p_{t+1}(1-\delta) - r_{t+1} \to \frac{p_t}{p_{t+1}} = 1 - \delta + \frac{r_{t+1}}{p_{t+1}} \quad (6)$$

From the firm's problem

$$\max_{k_t} p_t A k_t^\alpha n^{1-\alpha} - r_t k_t - w_t n$$

we have

$$r_t = p_t \alpha A k_t^{\alpha-1} \tag{7}$$

$$w_t = p_t(1-\alpha) A k_t^\alpha \tag{8}$$

From (3), (6), and (7) we get $\frac{c_{t+1}^i - \bar{c}}{c_t^i - \bar{c}} = \beta(1 - \delta + \alpha A k_{t+1}^{\alpha-1})$.

Now having these in mind, recall that in problem set 1, question 4, we showed the equivalence of time-0 trading and sequential markets. The notion of wealth that we developed there, is basically the value of assets $q_t a_t$ that each agent holds. Remember that in each period t the wealth of (i.e. the value of assets held by) an agent is

$$W_t^R = r_t k_t^R + w_t n^R - p_t \left(c_t^R + k_{t+1}^R - (1-\delta) k_t^R\right)$$
$$W_t^P = r_t k_t^P + w_t n^P - p_t \left(c_t^P + k_{t+1}^P - (1-\delta) k_t^P\right)$$

$$\begin{aligned} W_t^R - W_t^P &= (r_t + p_t(1-\delta))(k_t^R - k_t^P) - p_t(k_{t+1}^R - k_{t+1}^P) - p_t \left(c_t^R - c_t^P\right) \\ &= p_{t-1}(k_t^R - k_t^P) - p_t(k_{t+1}^R - k_{t+1}^P) - p_t \left(c_t^R - c_t^P\right) \\ &= p_t(1 - \delta + \alpha A k_{t+1}^{\alpha-1})(k_t^R - k_t^P) - p_t(k_{t+1}^R - k_{t+1}^P) - p_t \left(c_t^R - c_t^P\right) \end{aligned}$$

Let's assume that $k_0 < k_{ss}$, so that capital is converging upwards to the steady state. Then the term $\beta(A\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$ is 'large' at $t = 0$, resulting in the reach begin wealthier initially. However, in the steady state for each agent $r_t k_t^i + w_t n^i - p_t \left(c_t^i + k_{t+1}^i - (1-\delta) k_t^i\right) = 0$. Therefore the wealth inequality shrinks over time to 0.

Problem 7

For reference on this problem, see Phelps and Pollack (1968).

Part (a)

In this formulation, where the time 0 generation decides on all future consumption and investment, there is no inconsistency and we can solve the problem using the usual methods. Taking FOC:

$$c_0^{-1} = \lambda_0$$
$$\delta\beta^t c_t^{-1} = \lambda_t \ \forall\ t > 0$$
$$\lambda_t = R\lambda_{t+1}$$

Solving, we get

$$c_0 = \frac{c_1}{R\delta\beta}$$
$$c_t = \frac{c_{t+1}}{R\beta} \ \forall\ t > 0$$

Can we solve for c_t? Supposing that we had finite horizon T:

$$c_T = rk_T$$
$$c_{T-1} = \frac{Rk_{T-1}}{1+\beta}$$
$$c_{T-2} = \frac{Rk_{T-2}}{1+\beta+\beta^2}$$
$$...$$
$$c_t = (1-\beta)Rk_t$$

We can solve for c_0 by noting that since $c_1 = R\delta\beta c_0$, we have $k_1 = \frac{R\delta\beta}{R(1-\beta)}c_0$

$$c_0 + k_1 = Rk_0 \to c_0(1 + \frac{R\delta\beta}{R(1-\beta)}) = Rk_0$$

and hence

$$c_0 = \frac{1-\beta}{1-\beta(1-\delta)}Rk_0$$

Can we solve this using dynamic programming? Since the only 'special' period is the first period, and all subsequent periods are identical, we can divide the problem into two parts: the initial decision about k_1, and all future decisions. For periods $t > 0$, the problem can be solved recursively:

$$V(k; k_1) = \max_{k'} \left[\log(Rk - k') + \beta V(k'; k_1)\right]$$

and then the problem at time $t = 0$ becomes

$$U = \max_{k_1} [\log(Rk_0 - k_1) + \delta\beta V(k_1; k_1)] \tag{1}$$

Part (b)

With the addition of "t-selves", we introduce the problem of time-inconsistency. Now, the policy decision that maximizes the utility of the time 0 generation will come into conflict with the decision that maximizes the utility of any other generation. From the standpoint of 0-self, the optimal decision at time $t = 1$ is

$$\frac{1}{Rk_1 - k_2} = \beta V'(k_2; k_1) \tag{2}$$

whereas for 1-self, the optimal decision corresponds to (1) from above:

$$\frac{1}{Rk_1 - k_2} = \delta\beta V'(k_2; k_2) \tag{3}$$

The RHS of (3) is smaller than the RHS of (2), meaning that 1-self wants c_1 to be larger than would 0-self. Hence, if 0-self is allowed to dictate the savings policies for future generations, the utility of those generations will not be maximized.

Part (c)

The problem is now different: each generation takes future generations' savings decisions as given. The problem cannot be written as a Bellman because we still have hyperbolic discounting. However, we can write the problem as the choice of next-period capital k_{t+1}, given future savings behavior $g(k)$, that solves

$$\max_{k_{t+1}} \; [\log(Rk_t - k_{t+1}) + \delta V(k_{t+1})] \tag{4}$$

$$V(k_{t+1}) = \sum_{s=t+1}^{\infty} \beta^{s-t} \log[Rk_s - g(k_s)]$$

$$k_{t+1} \geq 0 \; \forall t$$

$$k_t \text{ given}$$

Part (d)

Suppose all future generations save at rate $g(k) = Ak$. Given this guess, the continuation value $V(k_{t+1})$ becomes

$$\begin{aligned} V(k_{t+1}) &= \sum_{s=t+1}^{\infty} \beta^{s-t} \log[(R-A)k_s] \\ &= \beta \left(\sum_{s=t+1}^{\infty} \beta^{s-t-1} \log[(R-A)A^{s-t-1}k_{t+1}] \right) \end{aligned}$$

Taking FOC of (4) w.r.t. k_{t+1},

$$\begin{aligned}
\frac{1}{Rk_t - k_{t+1}} &= \delta\beta \sum_{s=t+1}^{\infty} \left(\beta^{s-t-1} \frac{(R-A)A^{s-t-1}}{(R-A)A^{s-t-1}k_{t+1}} \right) \\
&= \delta\beta \sum_{s=t+1}^{\infty} \left(\beta^{s-t-1} k_{t+1}^{-1} \right) \\
&= \frac{\delta\beta}{(1-\beta)k_{t+1}}
\end{aligned}$$

Solving for k_{t+1}, we get

$$\begin{aligned}
k_{t+1} &= \frac{\delta\beta}{1-\beta(1-\delta)} Rk_t \\
&= Ak_t \\
\implies c_t &= \frac{1-\beta}{1-\beta(1-\delta)} Rk_t
\end{aligned}$$

which verifies the guess. Note that t-self's decision is independent of the future savings rate A - this comes from the assumption of log utility. This is also why the solution for this part coincides with the solution in part (a): the fact that 0-self takes into account future generations' savings policy doesn't really matter for today's savings decision, as long as that future policy is consistent.